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THE SECOND HANKEL DETERMINANT FOR A CLASS OF SPIRALLIKE FUNCTIONS DEFINED BY HOHLOV OPERATOR

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Abstract. By making use of the Hohlov operator given by the class of Spirallike functions is introduced. The object of the present paper is to obtain sharp upper bound for functional $|a_2a_4 - a_3^2|$.

1. Introduction, Definition and Motivation

Let A denotes the class of normalized analytic functions of the form,

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

where,

$$z \in \mathbb{E} \quad \{z : z \in \mathbb{C} \quad \& \quad |z| < 1\} \tag{1.2}$$

Let S denotes the class of all functions in A which are univalent.

Robertson [14] introduce to class of starlike function of order β as follows,

Definition 1.1. *Let $\delta \in [0,1]$, $f \in S$ &*

$$R\left\{\frac{zf'(z)}{f(z)}\right\} > \delta \quad z \in \mathbb{E} \tag{1.3}$$

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We say that f is starlike function of order β & denoted by S$^*(\beta)$. Spacek[16] introduce the class of Spirallike function of type β as follows,

Definition 1.2. *Let $f \in S$, $-\pi/2 < \beta < \pi/2$ then f(z) is spirallike function of type β on E.*

$$R\left\{\frac{e^{i\beta} z f'(z)}{f(z)}\right\} > 0 \quad z \in \mathbb{E} \tag{1.4}$$

denoted class of S_β.

From definition (1.1) & (1.2) it is easy to see [18] that Starlike functions of order β & Spirallike functions of type β have some relationship on geometry. Starlike functions of order β map E into the right half complex plane whose real part is greater than β by mapping $\frac{zf'(z)}{f(z)}$, while spirallike functions of type β map E into the right half complex plane by the mapping $\frac{e^{i\beta} z f'(z)}{f(z)}$. Since,

$$\lim_{z \to 0} \frac{e^{i\beta} z f'(z)}{f(z)} = e^{i\beta}$$

We can deducted that if we restrict the image of the mapping $\frac{e^{i\beta} z f'(z)}{f(z)}$ in the right complex plane whose real part is greater than a certain constant, then the constant must be smaller than $\cos\beta$. Libra [16] introduced & studied the class S$_\delta^\beta$ given as follows,

Definition 1.3. *Let $\delta \in$ [0,1], -$\pi/2 < \beta\pi/2$ & $f \in S$ then, $f \in S_\delta^\beta$ if and only if,*

$$R\left\{\frac{e^{i\beta} z f'(z)}{f(z)}\right\} > \delta \cos\beta \quad z \in \mathbb{E} \tag{1.5}$$

The qth determinant for q $\geq$ 1 and n $\geq$ 0 is stated by Noonan and Thomas [10] as,

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & & & \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}$$

This determinant has also been considered by several authors. For example, Noorin [20] determined the rate of growth of $H_q(n)$ as n $\to \infty$ for functions f given by (1.3) with bounded boundary. Ehrenborg [21] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman's article . It is well known that [1] for f $\in$ S and given by (1.5) the sharp inequality $|a_3 - a_2^2| \leq 1$ hold. This corresponds to the Hankel determinant with q = 2 and k = 1. After that, Fekete-Szego further generalized the estimate $|a_3 - \mu a_2^2|$ with real μ and f $\in$ S. For a given class of function in $\mathcal{A}$, the sharp bound for the non linear functional $|a_2 a_4 - a_3^2|$ is known as the second Hankel determinant. This corresponds to the Hankel determinant with q = 2 and k = 2.

For the function f & g $\in$ A given by the series,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \& \quad g(z) = \sum_{n=0}^{\infty} b_n z^n \quad z \in \mathbb{E} \tag{1.6}$$

The Hadamard product of f & g denoted by f $*$ g is defined as,

$$f * g = \sum_{n=0}^{\infty} a_n b_n z^n = (g * f)(z) \tag{1.7}$$

By using the Hadamard product hohlov [12] introduced and studied the linear operator $I_c^{a,b} : \Omega \to \Omega$ defined by,

$$I_c^{a,b} f(z) = \quad {}_2F_1(a,b;c;z) * f(z) \quad f \in \Omega \quad z \in \mathbb{E} \tag{1.8}$$

where ${}_2F_1$(z) known as Gaussian hypergeometric function is defined by

$$ {}_2F_1(z) = 2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n \tag{1.9}$$

where $(a, b \in \mathbb{C}, c \in \mathbb{C} \backslash z_{\bar{0}} = \{0, -1, -2, \ldots\})$

$\lambda_n$is the Pochhamer Symbol or Shifted factorial written in terms of Gamma Function Γ by,

$$ (\lambda_n) = \frac{\Gamma\lambda + n}{\Gamma n} = \begin{cases} 1 & n = 0 \\ \lambda(\lambda+1)(\lambda+2) & n \in \mathbb{N} = \{1, 2, \ldots\} \end{cases} \tag{1.10}$$

Note that ${}_2F_1$ is symmetric in a & b and that the series terminates if at least one of the numerator parameter a & b is zero or a negative integer. Observe that for the function f of the form (1.1) we have,

$$ I_c^{a,b} f(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (1)_{n-1}} a_n z^n \tag{1.11}$$

Definition 1.4 (16). *A function $f \in A$ is said to be in the class of $S_c^{a,b}$ (β, δ) $(|\beta| < \pi/2, 0 \leq \delta < 1)$ if it satisfies the inequality,*

$$ R\left\{ \frac{e^{i\beta} I_c^{a,b} f(z)}{z} \right\} > \delta \cos\beta \tag{1.12}$$

Definition 1.5. *Let P be the family of all functions p analytic in E for which, $R\{P(z)\} > 0$ & P(z) = 1 + C_1 + C_2 + ..., $z \in E$.*

$$ \begin{aligned} & f \in S_c^{a,b}(\beta, \delta) \\ & \Leftrightarrow e^{i\beta} \frac{\mathbf{I}_c^{a,b} f(z)}{z} \\ & = [(1-\delta)p(z) + \delta] \cos\beta + i \sin\beta \end{aligned} \tag{1.13}$$

where β is real, $|\beta| \leq \pi/2$ & p(z) $\in P$. We note that,

$$ S_1^{0,0}(\beta, \delta) = \left\{ f : f \in A \quad \& \quad R\left\{ e^{i\beta} \frac{f(z)}{f(z)} \right\} > \delta \cos\beta \right\} \tag{1.14}$$

$$S_1^{0,1}(\beta,\delta) = \Big\{ f : f \in A \quad \& \quad R\Big\{e^{i\beta} f'(z)\Big\} > \delta \cos\beta \Big\} \tag{1.15}$$

$$\begin{aligned} S_1^{0,1}(0,0) &= S_1^{1,0}(0,0) = S_2^{0,1}(0,0) \\ = R &= \Big\{ f : f \in A \quad \& \quad R\{f'(z)\} > 0 \Big\} \end{aligned} \tag{1.16}$$

Janteng, Halim and Darus [3] have considered the functional $|a_2a_4 - a_3^2|$ and found a sharp upper bound for the function f in the subclass RT of S consisting of function whose derivative has a positive real part studied by MacGregor [11]. In their work they have show that if f $\in$ RT then $|a_2a_4 - a_3^2| \le \frac{4}{9}$. Janteng et al obtain the second Hankel determinant and sharp upper bounds for the familiar subclass of S, namely starlike and convex functions denoted by ST & CV and showed that $|a_2a_4 - a_3^2| \le 1$ and $|a_2a_4 - a_3^2| \le \frac{1}{8}$ respectively.

Aabed Mohammed and Maslina Darus [1] for some recent work [2][3][4][5] obtained sharp upper bound to the second hankel determinant for the class of analytic function defined by linear operator. Motivated by the above mentioned results by different authors in this direction. In this paper we generalized the results by finding sharp upper bounds for $\mathrm{H}_2(2)$ for f in $\mathrm{S}_c^{a,b}$ (β, δ) defined by Hohlov Operator.

2. Preliminaries & Notations

Lemma 2.1. *If the function $p \in P$ is given by the series,*

$$p(z) = 1 + p_1 z + p_2 z^2 + \dots$$

then the following sharp estimate holds,

$$|p_k| \le 2 \qquad k = 1, 2, \dots$$

Lemma 2.2. *If the function $p \in P$ is given by the series then,*

$$2p_2 = p_1^2 + x(4 - p_1^2)$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z$$

for some x,z, $|x| \leq 1$, $|z| \leq 1$.

3. Main Results

Theorem 3.1. *Let the function f given by (1.1) be in the class $S_c^{a,b}$ (β, δ) then,*

$$|a_2a_4 - a_3^2| \leq \frac{16c^2(1-\delta)^2(c+1)^2\cos^2\beta}{a^2b^2(a+1)^2(b+1)^2}$$

Proof. Let f $\in$ $S_c^{a,b}$ (β, δ) then,

p $\in$ P is given by (1.12) then

$$e^{i\beta}\frac{I_c^{a,b}f(z)}{z} = \Big[[1-\delta]p(z) + \delta\Big](\cos\beta + i\sin\beta) \tag{3.1}$$

$$e^{i\beta}\left\{1 + \sum_{k=2}^{\infty}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(1)_{k-1}}a_kz^{k-1}\right\} = [(1-\delta)[1 + \sum_{k=1}^{\infty}p_kz^k] \\ +\delta](\cos\beta + i\sin\beta) \tag{3.2}$$

Comparing the coefficients, we get,

$$e^{i\beta}\frac{(a)(b)}{c}a_2 = (1-\delta)p_1\cos\beta \tag{3.3}$$

$$\therefore a_2 \times e^{i\beta} = \frac{c(1-\delta)p_1\cos\beta}{ab} \tag{3.4}$$

$$a_3 \times e^{i\beta} = \frac{2c(c+1)(1-\delta)p_2\cos\beta}{ab(a+1)(b+1)} \tag{3.5}$$

$$a_4 \times e^{i\beta} = \frac{6c(c+1)(c+2)(1-\delta)p_3\cos\beta}{ab(a+1)(a+2)(b+1)(b+2)} \tag{3.6}$$

$$\begin{aligned}
|a_2a_4 - a_3^2| &= \left| \frac{6c^2(1-\delta)^2 p_1 p_3 \cos^2\beta (c+1)(c+2)}{a^2b^2(a+1)(a+2)(b+1)(b+2)} \right. \\
&\qquad \left. - \frac{4c^2(c+1)^2(1-\delta)^2 p_2^2 \cos^2\beta}{a^2b^2(a+1)^2(b+1)^2} \right| \\
&= \frac{c^2(1-\delta)^2(c+1)cos^2\beta}{a^2b^2(a+1)(b+1)} \left| \frac{6(c+2)p_1p_3}{(a+2)(b+2)} \right. \\
&\qquad \left. - \frac{4(c+1)p_2^2}{(a+1)(b+1)} \right|
\end{aligned} \tag{3.7}$$

□

Since the function p(z) and p($e^{i\theta_2}$) ($\theta \in$ R) are members of the class p, simultaneously we assume without loss of generality $p_1 >$ 0 for convenience of notation. We take $p_1 =$ P, p $\in$ [0,2] by using lemma,

$$\begin{aligned}
|a_2a_4 - a_3^2| = \frac{c^2(1-\delta)^2(c+1)\cos^2\beta}{a^2b^2(a+1)(b+1)} &\left| \frac{6(c+2)}{(a+2)(b+2)} \left(\frac{p^4}{4} + \frac{(4-p^2)xp^2}{2} - \right. \right. \\
&\left. \frac{p^2(4-p^2)x^2}{4} + \frac{2p(4-p^2)}{4}(1-|x|^2z) \right) \\
&\left. - \frac{4(c+1)}{(a+1)(b+1)} \left[\frac{p^4 + 2p^2x(4-p^2) + x^2(4-p^2)^2}{4} \right] \right|
\end{aligned} \tag{3.8}$$

An application of triangle inequality and replacement of $|x|$ by y gives,

$$|a_2a_4 - a_3^2| \leq \frac{c^2(1-\delta)^2(c+1)\cos^2\beta}{a^2b^2(a+1)(b+1)}\left[\left\{\frac{6(c+2)}{(a+2)(b+2)}\right.\right.$$
$$\left.\left.\left(\frac{p^4}{4} + \frac{(4-p^2)p^2y}{2} + \frac{p^2(4-p^2)}{4}y^2 + \frac{p(4-p^2)}{2}(1-y^2)\right)\right\}\right] \quad (3.9)$$
$$+\frac{(c+1)}{(a+1)(b+1)}\left[p^4 + 2p^2y(4-p^2) + y^2(4-p^2)^2\right]$$
$$= G(p,y) \quad 0 \leq p \leq z \quad 0 \leq y \leq 1$$

We maximize the function G(p,y) on closed rectangle [0,2] $\times$ [0,1] since,

$$\frac{\delta G}{\delta y} = \frac{c^2(1-\delta)^2(c+1)\cos^2\beta}{a^2b^2(a+1)(b+1)}\left[\left(\frac{6(c+2)}{(a+2)(b+2)}\right)\frac{(4-p^2)2p}{2}\right.$$
$$\left. - \frac{2p^2y(4-p^2)}{4} - \frac{2y(4-p^2)p}{2}\right] + \frac{4(c+1)}{(a+1)(b+1)}$$
$$\left[\frac{2p^2(4-p^2) + 2y(4-p^2)^2}{4}\right]$$

$$G(p,1) = F(p)$$
$$= \frac{c^2(1-\delta)^2(c+1)\cos^2\beta}{a^2b^2(a+1)(b+1)}\left\{\left[\left(\frac{c+2}{(a+2)(b+2)}\right)\frac{p^2}{4} + \frac{3}{4}p^2(4-p^2)\right]\right.$$
$$\left. + \frac{4(c+1)}{(a+1)(b+1)}\left[\frac{p^4 + 2p^2(4-p^2) + (4-p^2)^2}{4}\right]\right\}$$

$$F'(p) = \frac{c^2(1-\delta)^2(c+1)\cos^2\beta}{a^2b^2(a+1)(b+1)}\left\{\left[\frac{6(c+2)}{(a+2)(b+2)}\left(\frac{2p}{4} + \frac{3(8p-4p^3)}{4}\right)\right.\right.$$
$$\left.\left. + \frac{4(c+1)}{(a+1)(b+1)}\left[\frac{4p^3 + 16p - 8p^3 + 2(4-p^2)2p}{4}\right]\right]\right\} \quad (3.10)$$

Where, $F'(p) < 0$, $0 < p < 2$, $p = 0$, $F(p) > F(2)$

$Max_{0 \leq p \leq 2}$ F(p) occurs at p = 0. $\therefore$ Upper bound (3.10) to y = 1, P = 0. Hence

$$|a_2a_4 - a_3^2| \leq \frac{16c^2(1-\delta)(c+1)^2\cos^2\beta}{a^2b^2(a+1)^2(b+1)^2} \tag{3.11}$$

Remark: For $\beta = 0$, $\delta = 0$, a = c = 1, b = 2, $|a_2a_4 - a_3^2| \leq \frac{4}{9}$. We get a recent result due to the Janteng et al. [3]

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References

[1] A. Mohammed and M. Darus, *Second Hankel determinant for a class of analytic function defined by Linear Operator*, Tamkang Journal of Mathematics, Vol. 43, No. 3, 455-462

[2] A. Janteng, S. A. Halim, and M. Darus, *Coefficient inequality for a function whose derivative has positive real part*, J.Ineq. Pure and Appl.Math.,7(2)(2006), 1-5.

[3] A. Janteng, S. A. Halim, and M. Darus, *Hankel determinant for starlike and convex functions*, Int. Journal of Math. Analysis,1(13)(2007), 619-625.

[4] A. K. Mishra and P. Gochhayat, *Second Hankel determinant for a class of ana lytic functions defined by fractional derivative functions*, International Journal of Mathematics and Mathematical Sciences, 2008, Article ID 153280, 10 pages.

[5] S. C. Soh and D.Mohamad, *Coefficient bounds for certain classes of closeŰtoŰconvex functions*, Int. Journal of Math. Analysis, 2(27)(2008), 1343-1351.

[6] G. Murugusundaramoorthy and N. Magesh, *Coefficient inequalities for certain classes of analytic functions associated with Hankel determinant*, Bulletin of Math. Anal. Appl., 1(3)(2009),85-89.

[7] R. J. Libera and E. J. Zlotkiewicz, *Early coefficients of the inverse of a regular convex function*, Proc. Amer. Math. Soc., 85(2)(1982), 225-230.

[8] R. J. Libera and E. J. Zlotkiewicz, *Coeffcient bounds for the inverse of a function with derivative in P*, Proc. Amer.Math. Soc., 87(2)(1983), 251-289.

[9] P. L. Duren, *Univalent Functions*, SpringerŰVerlag, New York, Berlin, Heidelberg, Tokyo, 1983.

[10] J. W. Noonan and D.K.Thomas, *On the second Hankel determinant of areally mean $p-$ valent functions*, Trans. Amer.Math. Soc., 223(2)(1976), 337-346.

[11] T. H.MacGregor,*Functions whose derivative has a positive real part*, Trans. Amer.Math. Soc., 104(1962), 532-537.

[12] Yu.E.Hohlov, *Operators and operations in the class of univalent functions (in Russian)*, Izv. Vyass. Ucebn. Zaved. Matematika 10(1978), 83-89.

[13] S. Goyal, R. K. Laddha, *On the generalized Riemann zeta function and the generalized Lambert transform*, Ganita Sandesh, 11(1997), 99-108.

[14] M.S. Robertson,*On the theory of univalent functions*, Ann.Math., 37(1936), 374-408.

[15] L. Spacek, *Contribution la thorie des fonctions univalentes*, Casopis Pest Math., 62(1932), 1219, (in Russian).

[16] J. R. Libera, *Univalent $\alpha-$ spiral functions*, Canada. J.Math., 19(1967), 449-456.

[17] Q. Xu and S. Lu, *The Alexander transformation of a subclass of spirallike functions of type β*, J. Ineq. Pure and Appl.Math, 10(1)(2009), 1-7.

[18] T. Hayami and S. Owa, *Hankel determinant for pŰvalently starlike and convex functions of order α*, General Math., 17 (2009), 29-44.

[19] T. Hayami and S. Owa, *Generalized Hankel determinant for certain classes*, Int. J.Math. Anal., 4 (2010), 2573-2585.

[20] K. I. Noor, *Hankel determinant problem for the class of functions with bounded boundary rotation*, Rev. Roum. Math. Pures Et Appl., 28(8) (1983), 731-739.

[21] R. Ehrenborg, *The Hankel determinant of exponential polynomials*, Amer. Math. Monthly, 107(6)(2000), 557-560.

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