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Common fixed point theorem on b-metric space

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Abstract: The purpose of this paper is to prove unique common fixed point theorem using weakly compatible mapping in a complete b-metric space.
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Introduction

In 1989, the concept of b-metric space was initiated by I.A.Bakhtin [3]. Czerwik was presented a generalization of banach fixed point theorem in the b-metric spaces. The concept of weakly compatible mappings was introduced by G.Junck and B.E Rhoades [5] in metric space.

1.1 Definition: Let X be a nonempty set and $s\geq 1$ be a given real number. A function d: $X \times X \to R^+$ is a b-metric if for all $x,y,z\in X$, the following conditions are satisfied.

(i) d(x,y)=0 iff x=y.
(ii) d(x,y)=d(y,x)
(iii) $d(x,z)\leq s[d(x,y)+d(y,z)]$

The pair (X,d) is called a b-metric space.

It is clear that b-metric space is effectively larger than that of metric spaces. If we consider s=1 in the definition-1 then we obtain the definition of usual metric space. So our results are more general than the same results in usual metric space.

The following example shows that in general b-metric space need not necessarily be a metric space.

1.2 Example[7]: Let (X,d) be a metric space and $\rho(x,y)=(d(x,y))^p$ where p>1 is a real number. Then ρ is a b-metric with $s=2^{p-1}$. Clearly the condition (i) and (ii) of definition1.1 are satisfied. If $1<p<\infty$, then convexity of the function $f(x)=x^p (x>0)$ implies that $\left(\frac{a+b}{2}\right)^p \leq \frac{1}{2}\left(a^p+b^p\right)$

i.e.

$(a+b)^p \leq 2^{p-1}\left(a^p+b^p\right)$. Thus for each $x,y,z\in X$ we have

$$\rho(x,y)= (d(x,y))^p\leq (d(x,z)+d(z,y))^p$$
$$\leq 2^{p-1}\left((d(x,z))^p+(d(z,y))^p\right)$$
$$=2^{p-1}\left(d(x,z)+d(z,y)\right)$$

So the condition (iii) of definition 1.1 is satisfied. Hence ρ is a b-metric space. However, if (X,d) is a metric space, then (X,ρ) is not necessarily a metric space. For example if X=R(set of real numbers) and d(x,y)= $|x-y|$ is the usual metric , then $\rho(x,y)=(x-y)^2$ is a b-metric on R with s=2, but is not a metric on R.

1.3 Definition: Two self maps S,T of a metric space (X,d) are said to be a compatible mappings of type-(A) if $\lim_{n\to\infty} d(STx_n,TTx_n)=\lim_{n\to\infty} d(TSx_n,SSx_n)=0.$

whenever $\{x_n\}$ is a sequence in X such that

$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_{n+1} = t$ for some t∈X

1.4 Definition: Let S and T be the two self maps defined on set X. Then S and T are said to be weakly compatible if they commute at every coincidence point.

1.5 Definition: Let $\{x_n\}$ be a sequence in a b-metric space (X,d).

(i) $\{x_n\}$ is called b-convergent if and only if there is x∈X such that $d(x_n, x) \to 0 \;\; as \;\; n \to \infty.$

(ii) $\{x_n\}$ is called b-Cauchy sequence if and only if there is x∈X such that $d(x_n, x_m) \to 0 \;\; as \;\; n,m \to \infty.$

A b-metric space is said to be complete if and only if each b-Cauchy sequence in this space is b-convergent.

The following theorem was proved by P.Sanodia and etal.[1],

2.1 Theorem-A: Let (X,d) be a complete b-metric space with constant s≥1 and S and T are two self mappings such that

(i) T(X) ⊆ S(X)
(ii) One of S or T be continuous
(iii) (S,T) is of compatible of type(A)
(iv)

$$d(Tx,Ty) \le a\,Max\begin{Bmatrix} d(Tx,Sy), d(Sx,Sy), \\ d(Ty,Sy), d(Tx,Sx) \end{Bmatrix} + b[d(Ty,Sx)]$$

where a+2sb≤1, ∀x,y∈X.

Then S and T have a unique common fixed point.

We are proving above theorem using the concept of weakly compatible mappings and without continuity.

2.2 Theorem-B: Let (X,d) be a complete b-metric space with constant s≥1 and S and T are two self mappings such that

(i) T(X) ⊆ S(X)
(ii) S(X) or T(X) be closed subspace of X.
(iii) The pair (S,T) is weakly compatible
(iv)

$$d(Tx,Ty) \le \; a.\; Max\begin{Bmatrix} d(Tx,Sy), d(Sx,Sy), \\ d(Ty,Sy), d(Tx,Sx) \end{Bmatrix} + b[d(Ty,Sx)]$$

where a+2sb≤1, b>0, a∈(0,1) ∀x,y∈X.

Then S and T have a unique common fixed point.

Proof: Let x_0∈X, T(X) ⊆ S(X) then there exist x_{n+1} and x_n in X such that $Tx_n=Sx_{n+1}$, n=0,1,2,3………… Now put $x=x_n$ and $y=x_{n+1}$ in equation (iv) of Theorem –B, we get

$$d(Tx_n, Tx_{n+1}) \le \; a.\; Max\begin{Bmatrix} d(Tx_n, Sx_{n+1}), d(Sx_n, Sx_{n+1}), \\ d(Tx_{n+1}, Sx_{n+1}), d(Tx_n, Sx_n) \end{Bmatrix} + b[d(Tx_{n+1}, Sx_n)]$$

$$d(Sx_{n+1}, Sx_{n+2})$$
$$\le a.Max\begin{Bmatrix} d(Tx_n, Sx_{n+1}), d(Sx_n, Sx_{n+1}), \\ d(Tx_{n+1}, Sx_{n+1}), d(Tx_n, Sx_n) \end{Bmatrix} + b[d(Tx_{n+1}, Sx_n)]$$
$$= a.Max\begin{Bmatrix} d(Sx_{n+1}, Sx_{n+1}), d(Sx_n, Sx_{n+1}), \\ d(Sx_{n+2}, Sx_{n+1}), d(Sx_{n+1}, Sx_n) \end{Bmatrix} + b[d(Sx_{n+2}, Sx_n)]$$
$$= a.Max\{0, d(Sx_n, Sx_{n+1}), d(Sx_{n+2}, Sx_{n+1}), 0\} + b[d(Sx_{n+2}, Sx_n)]$$
$$\le a.Max\{d(Sx_n, Sx_{n+1}), d(Sx_{n+2}, Sx_{n+1})\} + bs[d(Sx_{n+2}, Sx_{n+1}) + d(Sx_n, Sx_{n+1})]$$

Case-(i): If $d(Sx_n, Sx_{n+1}) > d(Sx_{n+2}, Sx_{n+1})$, then

$$d(Sx_{n+1}, Sx_{n+2}) \le a.Max\{d(Sx_n, Sx_{n+1})\} + bs[d(Sx_{n+2}, Sx_{n+1}) + d(Sx_n, Sx_{n+1})]$$

$$(1-sb)\,d(Sx_{n+1}, Sx_{n+2}) \le (a+sb)\{d(Sx_n, Sx_{n+1})\}$$
$$d(Sx_{n+1}, Sx_{n+2}) \le \left(\frac{a+sb}{1-sb}\right) d(Sx_n, Sx_{n+1})$$

$$d(Sx_{n+1}, Sx_{n+2}) \le k_1\, d(Sx_n, Sx_{n+1})$$

$$where\; k_1 = \left(\frac{a+sb}{1-sb}\right) < 1 \quad \_\_\_\_(2.2.1)$$

Case-(ii): Suppose, If $d(Sx_{n+1},Sx_{n+2})>d(Sx_n,Sx_{n+1})$, then

$$d(Sx_{n+1},Sx_{n+2})\le a.\{d(Sx_{n+1},Sx_{n+2})\} + bs[d(Sx_{n+2},Sx_{n+1})+d(Sx_n,Sx_{n+1})]$$

$$(1-a-sb)\,d(Sx_{n+1},Sx_{n+2})\le sb\ \ d(Sx_n,Sx_{n+1})$$

$$d(Sx_{n+1},Sx_{n+2})\le\left(\frac{sb}{1-a-sb}\right)d(Sx_n,Sx_{n+1})$$

$$where\ k_2=\left(\frac{sb}{1-a-sb}\right)<1$$

$$d(Sx_{n+1},Sx_{n+2})\le k_1\,d(Sx_n,Sx_{n+1})$$

$$where\ k_1=\left(\frac{a+sb}{1-sb}\right)<1 \_\_\_\_\_\_ (2.2.2)$$

By the equations (2.2.1) and (2.2.2), Let k= max.{k_1, k_2}, since $k_1<1$ and $k_2<1$ gives k<1.

$$d(Sx_{n+1},Sx_{n+2})\le k\,d(Sx_n,Sx_{n+1})$$
$$d(Sx_{n+1},Sx_{n+2})\le k^2\,d(Sx_{n-1},Sx_n)$$
$$d(Sx_{n+1},Sx_{n+2})\le k^3\,d(Sx_{n-2},Sx_{n-1})$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$d(Sx_{n+1},Sx_{n+2})\le k^n\,d(Sx_0,Sx_1)$$

Now we show that $\{Sx_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Let m,n>0, with m>n.

$$\begin{aligned}
d(Sx_n,Sx_m)&\le s[d(Sx_n,Sx_{n+1})+d(Sx_{n+1},Sx_m)]\\
&\le s\,d(Sx_n,Sx_{n+1})+s^2d(Sx_{n+1},Sx_{n+2})+s^2d(Sx_{n+2},Sx_m)\\
&\le s\,d(Sx_n,Sx_{n+1})+s^2d(Sx_{n+1},Sx_{n+2})+s^3d(Sx_{n+2},Sx_{n+3})+\ldots\\
&\le s\,k^n\,d(Sx_0,Sx_1)+s^2k^{n+1}d(Sx_0,Sx_1)+s^3k^{n+2}d(Sx_0,Sx_1)+\ldots\\
&\le s\,k^n\,d(Sx_0,Sx_1)\{1+sk+(sk)^2+(sk)^3+\ldots\}\\
&\le\frac{sk^n}{1-sk}\,d(Sx_0,Sx_1)
\end{aligned}$$

When we take m,n→∞, we get $\lim_{n\to\infty}d(Sx_n,Sx_m)=0$.

Hence $\{Sx_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

(X,d) is a complete metric space then sequence $\{Sx_n\}_{n=1}^{\infty}$ converges to some z in X. From the condition (i) of Theorem 2.2, we get subsequences Tx_n, Sx_{n+1} converges to z∈X.

Since either S(X) or T(X) is closed for definiteness consider S(X) is closed subspace of X. Then there exist for some u∈X such that $\lim_{n\to\infty}Tx_n=\lim_{n\to\infty}Sx_{n+1}=S(u)=z.$

Now to show T(u)=z, put x=u and y=x_n in condition (iv) of 2.2.

$$d(Tu,Tx_n)\le a\,Max\begin{Bmatrix}d(Tu,Sx_n),d(Su,Sx_n),\\ d(Tx_n,Sx_n),d(Tu,Su)\end{Bmatrix} + b\,[d(Tx_n,Su)]$$

$$d(Tu,z)\le\ a\ Max\begin{Bmatrix}d(Tu,z),d(z,z),\\ d(z,z),d(Tu,z)\end{Bmatrix}+b[d(z,z)]\}$$

$$d(Tu,z)\le\ a\ \ d(Tu,z)$$
$$(1-a)\ d(Tu,z)\le 0$$

This implies Tu=z, since (1-a)>0.

Therefore Su=Tu=z.

Since the pair (S,T) is weakly compatible, STu=TSu this implies Sz=Tz.

To show Tz=z, put x=z and y=u in the condition (iv) of theorem 2.2, we get

$$d(Tz,Tu)\le a.Max\begin{Bmatrix}d(Tz,Su),d(Sz,Su),\\ d(Tu,Su),d(Tz,Sz)\end{Bmatrix} + b\,[d(Tu,Sz)]$$

$$d(Tz,z)\le a.Max\begin{Bmatrix}d(Tz,z),d(Tz,z),\\ d(z,z),d(Tz,Tz)\end{Bmatrix}+b[d(z,Tz)]$$

$$d(Tz,z)\le(a+b)\,d(Tz,z)$$
$$[1-a-b]\ \ d(Tz,z)\le 0$$

This implies Tz=z, since [1-a-b] >0.

Hence Sz=Tz=z. Thus z is common fixed point of self maps S and T.

Uniqueness:

Let w be another fixed point of S and T, then Sw=Tw=w.

Put x=z and y=w in the condition (iv) of theorem 2.2, we get

$$d(Tz,Tw) \le a\,Max\begin{Bmatrix} d(Tz,Sw), d(Sz,Sw), \\ d(Tw,Sw), d(Tw,Sw) \end{Bmatrix} + b[d(Tw,Sw)]$$

$$d(z,w) \le a\,Max\begin{Bmatrix} d(z,w), d(z,w), \\ d(w,w), d(w,w) \end{Bmatrix} + b\,[d(w,w)]$$

$$d(z,w) \le a.\,d(z,w)$$

$$(1-a)\,d(z,w) \le 0$$

$$\Rightarrow z = w$$

This shows z is unique common fixed point of S and T.

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